

ANNIHILATORS OF IRREDUCIBLE MODULES AND KINEMATICAL CONSTRAINTS OF PAIR OPERATORS

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Abstract

The kinematical constraints of pair operators in nuclear collective motion, pointed out by Yamamura and identified by Nishiyama as relations between $so(2n)$ generators, are recognized as equations satisfied by second-degree annihilators (deduced in previous work) of irreducible $so(2n)$ -modules. The recursion relations for Nishiyama's tensors and their dependence on the parity of the tensor degree is explained. An explanation is also given for the recursion relations for $sp(2n)$ tensors pointed out by Hwa and Nuyts. The statements for the algebras $so(2n)$ and $sp(2n)$ are proved simultaneously.

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1 Introduction

In his algebraic approach to the theory of nuclear collective motion, Yamamura [1] pointed out a number of polynomial kinematical constraints that have to be satisfied by the pair operators (which are defined as commutators of fermions).

Observing that the pair operators generate the Lie algebra $so(2n)$, Nishiyama derived [2] these constraints by using the commutation relations of the generators of this algebra. He constructed, recurrently, sets of homogeneous polynomials of increasing degrees in the $so(2n)$ generators, each set possessing a specific $so(2n)$ tensor property which depends only on the parity of the degree of the polynomials which compose the set. The second-degree polynomials and the recursion relations are then used in [2] to derive Yamamura's kinematical constraints, whose Lie algebraic nature is established in this way.

Similar recursion relations had been pointed out by Hwa and Nuyts in their construction of $sp(2n)$ generators as anticommutators of boson operators [3]. Their study had, however, not the aim to investigate any identity satisfied by the $sp(2n)$ generators.

Let Λ_i , ($i = 1, \dots, n$) be the fundamental weights of a Lie algebra of rank n .

The present paper points out that Nishiyama's polynomials of second degree are precisely the irreducible tensors of highest weight $2\Lambda_1$ in the universal enveloping algebra $U(so(2n))$ of the $so(2n)$ algebra, whose explicit expressions have been derived in previous work [4]-[6] by projection from a generic second-degree element. These tensors were proved [6] to be annihilators of $so(2n)$ -modules which transform under (spinorial) representations of highest weights $k\Lambda_{n-1}$ and $k\Lambda_n$ of $so(2n)$.

As it is known, (e.g. [7]), the spinorial representations of $so(2n)$ can be constructed in terms of the generators of the Clifford algebra (in other words, in terms of creation and annihilation operators which anticommute); Nishiyama's result, which has been derived in a constructive way, appears thus as a consequence of the property of the $(2\Lambda_1)$ tensors in $U(so(2n))$ to be annihilators of spinorial modules.

The present paper proves also in a simple way that the tensors constructed by recursion in [2] belong to irreducible representations (IRs) that depend only on the parity of the tensor degree. The similar property observed in [3] for $sp(2n)$ is also proved.

Simultaneous proofs have been given for this property for the $so(2n)$ and $sp(2n)$ algebras. Indeed, as the $sp(2n)$ ($so(2n)$) generators can be constructed as anti-commutators (commutators) of boson (fermion) operators, the two cases can be distinguished only by the value of a parameter ϵ ($\epsilon = +1$ for $so(2n)$ and $\epsilon = -1$ for $sp(2n)$): we obtain in this way analogous expressions for the irreducible tensors and a simultaneous proof for the recurrence relation.

The identification of Nishiyama's kinematical constraints as second-degree tensors in the universal enveloping algebra stresses the physical relevance of the equations which result from the vanishing of these tensors, whose importance for physics has been already pointed out in [5],[6],[8]-[11].

2 Irreducible tensors in $U(L)$ and $S(L)$

Physicists knew since a long time that specific realizations of a Lie algebra L satisfy specific sets of relations (for a partial list of references cf. [6]) each relation consisting in the vanishing of a well-defined homogeneous polynomial whose indeterminates are the generators of the realization.

As observed in [4],[6] a set of such polynomials has specific tensorial properties under the adjoint action.

A realization ρ of a Lie algebra L is a homomorphism $\rho : L \rightarrow A$ of L into an associative algebra A endowed with a Lie bracket $[\cdot, \cdot]_A$, compatible with associativity. The homomorphism ρ can be extended to a homomorphism of associative algebras $\rho_U : U(L) \rightarrow A$ or $\rho_S : S(L) \rightarrow A$ (where $U(L)$ and $S(L)$ are the universal enveloping algebra of L and the symmetric algebra of L , respectively); extensions of the adjoint action of L to $S(L)$ and to $U(L)$ can be also defined, as well as irreducible tensors under this extension.

The irreducible tensors T_Λ (of highest weight Λ) of the extension of adL to $U(L)$ transform under subrepresentations (Λ) of the symmetric Kronecker powers $(ad^{\otimes k})_{sym}$ of adL . (This is a consequence of the property of the symmetrization operator to intertwine tensors in $S(L)$ and $U(L)$). As pointed out elsewhere [6], the tensors T_Λ vanish on specific irreducible representations (λ) of L , i.e. T_Λ are annihilators of the L -module (λ) .

For classical semisimple Lie algebras all second-degree tensors T_Λ in $S(L)$ that transform under a subrepresentation (of highest weight Λ) of the symmetric part $(ad \otimes ad)_{sym}$ of the Kronecker square of the adjoint representation have been determined [4]: the corresponding tensors T_Λ in the second-degree component $U^2(L)$ of $U(L)$ result from the tensors in $S^2(L)$ by symmetrization with respect to the order of the factors in each product.

For the algebras A_n, B_n, C_n and D_n all finite-dimensional L -modules (λ) which are annihilated by irreducible second-degree tensors T_Λ in $U(L)$ have been determined in [6] for all second-degree symmetric tensors T_Λ .

The irreducible tensors T_Λ into which decomposes the second-degree component $S^{(2)}$ of $S(L)$ result by projection $P_i S_{\alpha\beta} S_{\gamma\delta}$ from a generic element $S_{\alpha\beta} S_{\gamma\delta} \in S^2(L)$, where P_i is the projection operator associated with the eigenvalue c_i of the Casimir operator C of the IR Λ :

$$P_i = \frac{\prod_{j \neq i} (C - c_j I)}{\prod_{j \neq i} (c_j - c_i)} \quad (1)$$

This calculus will be performed simultaneously for the algebras $sp(2n, C)$ and $o(2n, C)$, for which identical ϵ -dependent expressions can be derived for the generators, the Casimir elements, the projection operators and, finally, for the irreducible tensors.

The generators of $\mathfrak{o}(2n, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$

Let us remind some well-known results.

The $\mathfrak{o}(2n, \mathbb{C})$ ($\mathfrak{sp}(2n, \mathbb{C})$) Lie algebra is defined as the set of linear operators Γ , acting in a $2n$ -dimensional complex vector space, which leaves invariant a non-degenerate symmetric (antisymmetric) bilinear form $(u, v) = \epsilon(v, u)$ with $\epsilon = +1$ ($\epsilon = -1$) for the orthogonal (symplectic) algebra; the invariance of (u, v) under an element Γ of the Lie algebra means that $(\Gamma u, v) + (u, \Gamma v) = 0$

In order to treat the $\mathfrak{o}(2n, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$ symmetry in a similar way one chooses for the symmetric bilinear form the expression

$$(u, v) = u^t K v \quad (2)$$

where X^t means the transpose of X and K is the $2n \times 2n$ matrix

$$K = \begin{pmatrix} 0 & I_n \\ \epsilon I_n & 0 \end{pmatrix} \quad (3)$$

(I_n is the n -dimensional unit matrix). Introducing the coordinates u_j, v_j ($j = 1, \dots, 2n$), the bilinear form (u, v) becomes

$$(u, v) = \sum_{i=1}^n (u_i v_{i+n} + \epsilon u_{i+n} v_i) \quad (4)$$

The invariance property of the bilinear form (u, v) leads to the following unique condition to be satisfied by the matrices of the orthogonal and symplectic algebras

$$\Gamma^t K + K \Gamma = 0 \quad (5)$$

with K given by (3); decomposing the matrix Γ into $n \times n$ blocks

$$\Gamma = \begin{pmatrix} A & B \\ -C & D \end{pmatrix} \quad (6)$$

we get from Eq. (5) the following conditions for the $n \times n$ matrices A, B, C and D :

$$B = -\epsilon B^t, \quad C = -\epsilon C^t, \quad D = -A^t \quad (7)$$

The generators Γ of the $\mathfrak{o}(2n, \mathbb{C})$ and $\mathfrak{sp}(2n, \mathbb{C})$ algebras are thus expressed as linear combinations of the following basic elements:

$$e_{ij} - e_{j+n, i+n} \quad \text{for the generators belonging to } A \oplus (-A^t) \quad (8)$$

$$e_{i, j+n} - \epsilon e_{j, i+n} \quad \text{for the generators belonging to } B \quad (9)$$

$$e_{i+n, j} - \epsilon e_{j+n, i} \quad \text{for the generators belonging to } -C \quad (10)$$

where $i, j = 1, \dots, n$ and $e_{\alpha\beta}$ are elements of the Weyl basis $(e_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}$ with $1 \leq \gamma, \delta \leq 2n$.

The generators of the two algebras can be written in the following form valid for both algebras

$$S_{\alpha\beta} = \sum_{\lambda=1}^{2n} (g_{\alpha\lambda}e_{\lambda\beta} - g_{\lambda\beta}e_{\lambda\alpha}) \quad (\alpha, \beta = 1, \dots, 2n) \quad (11)$$

where

$$g_{\alpha\beta} = \delta_{\alpha,\beta+n} + \epsilon\delta_{\alpha+n,\beta} \quad , \quad g_{\alpha\beta} = \epsilon g_{\beta\alpha} \quad , \quad S_{\alpha\beta} = -\epsilon S_{\beta\alpha} \quad (12)$$

Let us observe that $g \equiv (g_{\alpha,\beta}) = K^{-1}$ and that $g^{\alpha\beta} \equiv (g^{-1})_{\alpha\beta} = \epsilon g_{\alpha\beta}$.

With the notations (11)-(12), the commutation relations have then the same expression for both algebras, namely

$$[S_{\alpha\beta}, S_{\gamma\delta}] = g_{\gamma\beta}S_{\alpha\delta} + g_{\delta\alpha}S_{\beta\gamma} - g_{\alpha\gamma}S_{\beta\delta} - g_{\beta\delta}S_{\alpha\gamma} \quad (\alpha, \beta, \gamma, \delta = 1, \dots, 2n) \quad (13)$$

The $n \times n$ blocks A, B, C, D are recovered from Eqs.(11) and (12) by reintroducing the labels $i, j = 1, \dots, n$; we get:

$$S_{ij} = \epsilon e_{i+n,j} - e_{j+n,i} = -\epsilon S_{ji} \quad (14)$$

$$S_{i+n,j+n} = e_{i,j+n} - \epsilon e_{j,i+n} = -\epsilon S_{j+n,i+n} \quad (15)$$

$$S_{i+n,j} = e_{i,j} - e_{j+n,i+n} = -\epsilon S_{j,i+n} \quad (16)$$

and comparison with the expressions (8-10) leads to the identifications:

$$S_{i+n,j} = (A \oplus (-A^t))_{i,j} \quad (17)$$

$$S_{i+n,j+n} = B_{i,j} \quad (18)$$

$$S_{i,j} = -\epsilon C_{i,j} \quad (19)$$

Denoting, for simplicity, by A the sum of the matrices $A \oplus -A^t$, i.e denoting $A_{i,j} = S_{i+n,j}$, the unique commutation relation (13) is equivalent with the following set of distinct commutation relations ($i, j, k, l \leq n$):

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} - \delta_{il}A_{kj} \quad (20)$$

$$[B_{ij}, B_{kl}] = [C_{ij}, C_{kl}] = 0 \quad (21)$$

$$[A_{ij}, B_{kl}] = \delta_{jk}B_{il} - \epsilon\delta_{jl}B_{ik} \quad (22)$$

$$[A_{ij}, C_{kl}] = \epsilon\delta_{il}C_{jk} - \delta_{ik}C_{jl} \quad (23)$$

$$[B_{ij}, C_{kl}] = -\delta_{jk}A_{il} - \delta_{il}A_{jk} + \epsilon\delta_{ik}A_{jl} + \epsilon\delta_{jl}A_{ik} \quad (24)$$

Killing-Cartan bilinear form and dual elements..

The expression (11) of the generators leads to the following expression of the Killing-Cartan form valid both for $so(2n, \mathbb{C})$ and $sp(2n, \mathbb{C})$

$$(S_{\alpha\beta}, S_{\gamma\delta}) = tr(adS_{\alpha\beta}adS_{\gamma\delta}) = 8n(g_{\alpha\delta}g_{\gamma\beta} - \epsilon g_{\alpha\gamma}g_{\beta\delta}) \quad (25)$$

i.e. taking into account Eqs.(13)

$$(S_{\alpha\beta}, S_{\gamma\delta}) = 8n[(\delta_{\alpha,\delta+n} + \epsilon\delta_{\alpha+n,\delta})(\delta_{\gamma,\beta+n} + \epsilon\delta_{\gamma+n,\beta}) - (\delta_{\alpha,\gamma+n} + \epsilon\delta_{\alpha+n,\gamma})(\delta_{\beta,\delta+n} + \epsilon\delta_{\beta+n,\delta})] \quad (26)$$

whence the following pairs of elements which are dual with respect to the Killing-Cartan form can be determined:

$$(S_{\alpha \leq n, \beta \leq n}, \epsilon S_{\beta+n, \alpha+n}), \quad (S_{\alpha > n, \beta > n}, \epsilon S_{\beta-n, \alpha-n}), \quad (27)$$

$$(S_{\alpha \leq n, \beta > n}, S_{\beta-n, \alpha+n}), \quad (S_{\alpha > n, \beta \leq n}, S_{\beta+n, \alpha-n}) \quad (28)$$

or, using the notation introduced by the decomposition in blocks:

$$dual\ of\ B_{ij} = \epsilon C_{ji}, \quad dual\ of\ C_{ij} = \epsilon B_{ji}, \quad dual\ of\ A_{ij} = A_{ji} \quad (29)$$

Casimir element and projection operators

Let $\rho : L \rightarrow gl(V)$ be a representation of the Lie algebra L and β a bilinear symmetric form associated with ρ ; let x_i and x^i ($i = 1, \dots, dim L$) be dual elements of L with respect to ρ . The *Casimir element* of the representation ρ

$$C_\rho(\beta) = \sum_{i=1}^{dim L} \rho(x_i)\rho(x^i) \quad (30)$$

commutes with ρ and is basis independent. The action of the Casimir element $C = C_{ad}$ of the adjoint representation on the element $S_{\alpha\beta}$ writes

$$CS_{\alpha\beta} = \sum_{\kappa, \lambda, \mu, \nu=1}^{2n} g_{\kappa\nu}g_{\mu\lambda}[S_{\kappa\lambda}, [S_{\mu\nu}, \cdot]] = 8(n - \epsilon)S_{\alpha\beta} \quad . \text{ Using the identity}$$

$$[A, [B, CD]] = [A, C][B, D] + [B, C][A, D] + C[A, [B, D]] + [A, [B, C]]D$$

we write the action $C^k S_{\alpha\beta} S_{\gamma\delta}$ of the powers C^k of the Casimir element on the product $S_{\alpha,\beta} S_{\gamma,\delta}$ of two generic elements and calculate the expression of the projection operator (1).

The action of the projection operators P_{Λ_2} for $L = sp(2n)$, ($\epsilon = -1$) and ($P_{2\Lambda_1}$) for $L = o(2n)$, ($\epsilon = +1$), on the product $S_{\alpha\beta} S_{\gamma\delta}$ gives tensors of highest weights Λ_2 and $2\Lambda_1$, respectively:

$$P_{\Lambda_2(2\Lambda_1)} S_{\alpha\beta} S_{\gamma\delta} = \frac{C(C - 8(n - 2\epsilon))(C - 4(2n - \epsilon))}{(-4n)(8n - 16\epsilon - 4n)(8n - 4\epsilon - 4n)} S_{\alpha\beta} S_{\gamma\delta} =$$

$$\begin{aligned} & \frac{1}{2n-2\epsilon} [g_{\alpha\gamma}(T_{\beta\delta} - \frac{1}{2n}g_{\beta\delta}I_2) + g_{\beta\delta}(T_{\alpha\gamma} - \frac{1}{2n}g_{\alpha\gamma}I_2) - \\ & \epsilon g_{\alpha\delta}(T_{\beta\gamma} - \frac{1}{2n}g_{\beta\gamma}I_2) - \epsilon g_{\beta\gamma}(T_{\alpha\delta} - \frac{1}{2n}g_{\alpha\delta}I_2)] \end{aligned} \quad (31)$$

where

$$T_{\alpha\beta} = S_{\alpha\lambda}g^{\lambda\mu}S_{\mu\beta} \quad (32)$$

and

$$I_2 = g_{\alpha\beta}g_{\gamma\delta}S_{\alpha\delta}S_{\gamma\beta} \quad (33)$$

is the second-degree invariant. $T_{\alpha\beta}$ are components of the irreducible tensor in $S(L)$ with highest weight Λ_2 for $sp(2n)$ and $2\Lambda_1$ for $o(2n)$.

The vanishing of the elementary components in Eq.(31) lead to the equations:

$$E_{\alpha,\beta} \equiv T_{\alpha\beta} - \frac{1}{2n}g_{\alpha\beta}I_2 = 0 \quad (34)$$

Reminding the decomposition (7) in $n \times n$ blocks and the defining equations (11),(17-19), we obtain that Eq.(34) is equivalent with the following four sets of equations (in which $i, j = 1, 2, \dots, n$) :

$$E_{i,j} \equiv T_{ij} = (AB - BA^t)_{ij} = 0 \quad (35)$$

$$E_{i+n,j+n} \equiv T_{i+n,j+n} = -(CA - A^tC)_{ij} = 0 \quad (36)$$

$$E_{i,j+n} \equiv (A^2 - BC)_{ij} - \frac{\delta_{ij}}{2n}I_2 = 0 \quad (37)$$

$$E_{i+n,j} \equiv \epsilon((A^t)^2 - CB)_{ij} - \frac{\delta_{ij}}{2n}I_2 = 0 \quad (38)$$

where the expression of the Casimir invariant I_2 is:

$$I_2 = tr(A^2 - BC + (A^t)^2 - CB) \quad (39)$$

The corresponding tensors in $U(L)$ are obtained by symmetrization of Eqs.(35-38) with respect to order:

$$(AB - BA^t)_{ij} + \epsilon(AB - BA^t)^t_{ij} = 0 \quad (40)$$

$$(CA - A^tC)_{ij} + \epsilon(CA - A^tC)^t_{ij} = 0 \quad (41)$$

$$(A^2 - BC)_{ij} + ((A^t)^2 - CB)^t_{ij} = 2\frac{\delta_{ij}}{2n}I_2 \quad (42)$$

It has been proved [6] that these equations are verified by the generators of the IRs $(m\Lambda_n)$ for $sp(2n)$ and $(m\Lambda_{n-1})$, $(m\Lambda_n)$ for $so(2n)$, i.e. that the components of these tensors are annihilators of the irreducible modules $(m\Lambda_n)$ for $sp(2n)$ and $(m\Lambda_{n-1})$, $(m\Lambda_n)$ for $o(2n)$; for the $o(2n)$ algebra the corresponding IRs are spinorial.

3 The Hwa-Nuyts-Nishiyama (HNN) realisations

We shall treat the HNN realisations and recursion relations on an equal footing and consider therefore a Fock space F_ϵ for a system with n degrees of freedom on which representations of the canonical commutation ($\epsilon = -1$) or anticommutation ($\epsilon = +1$) relations are defined:

$$[b_i, b_j^+]_\epsilon \equiv b_i b_j^+ + \epsilon b_j^+ b_i = \delta_{ij} I \quad (43)$$

$$[b_i, b_j]_\epsilon = [b_i^+, b_j^+] = \delta_{ij} I \quad (i, j = 1, \dots, n) \quad (44)$$

Realisations of $sp(2n, R)$ have been defined on F_{-1} by Hwa and Nuyts [3]; realisations of $o(2n, R)$ have been defined on F_{+1} by Nishiyama [2]. Both realisations can be written as

$$E_j^i \equiv \frac{1}{2} [b_i^+, b_j]_{-\epsilon} = \frac{1}{2} (b_i^+ b_j - \epsilon b_j b_i^+) = b_i^+ b_j - \frac{\epsilon}{2} \delta_{ij} I \quad (45)$$

$$E_0^{ij} = \frac{1}{2} [b_i^+, b_j^+]_{-\epsilon} = \frac{1}{2} (b_i^+ b_j^+ - \epsilon b_j^+ b_i^+) = b_i^+ b_j^+ \quad (46)$$

$$E_{ij}^0 = \frac{\epsilon}{2} [b_i, b_j]_{-\epsilon} = \frac{\epsilon}{2} (b_i b_j - \epsilon b_j b_i) = \epsilon b_i b_j \quad (47)$$

The commutation relations of the generators (45-47) (pointed out in the Refs.[2], [3]) are precisely the commutation relations (20-24) if the following identifications are made:

$$A_{ij} = E_j^i, \quad B_{ij} = E_0^{ij}, \quad C_{ij} = -E_{ij}^0 \quad (48)$$

The HNN recursive relations

In the HNN papers series of operators defined by recursion have been introduced, these series can be described by the following unique formulas:

$${}^0 K_\beta^\alpha \equiv \delta_\beta^\alpha, \quad {}^0 K^{\alpha\beta} \equiv 0, \quad {}^0 K_{\alpha\beta} \equiv 0 \quad (49)$$

$${}^1 K_\beta^\alpha \equiv E_\beta^\alpha, \quad {}^1 K^{\alpha\beta} \equiv E^{\alpha\beta}, \quad {}^1 K_{\alpha\beta} \equiv E_{\alpha\beta}^0 \quad (50)$$

$${}^{m+1} K_\beta^\alpha \equiv \frac{1}{4} ([{}^m K_\tau^\alpha, E_\beta^\tau]_+ + [{}^m K_\beta^\tau, E_\tau^\alpha]_+ + [{}^m K^{\alpha\tau}, E_{\tau\beta}^0]_+ + [{}^m K_{\beta\tau}, E_0^{\tau\alpha}]_+) \quad (51)$$

$${}^{m+1} K^{\alpha\beta} \equiv \frac{1}{4} ([{}^m K_\tau^\alpha, E_0^{\tau\beta}]_+ - \epsilon(-)^m [{}^m K_\tau^\beta, E_0^{\tau\alpha}]_+ - [{}^m K^{\alpha\tau}, E_\tau^\beta]_+ + [{}^m K^{\tau\beta}, E_\tau^\alpha]_+) \quad (52)$$

$${}^{m+1} K_{\alpha\beta} \equiv \frac{1}{4} ([{}^m K_\alpha^\tau, E_{\tau\beta}^0]_+ - \epsilon(-)^m [{}^m K_\beta^\tau, E_{\tau\alpha}^0]_+ - [{}^m K_{\alpha\tau}, E_\beta^\tau]_+ + [{}^m K_{\tau\beta}, E_\alpha^\tau]_+) \quad (53)$$

with the symmetry properties

$${}^m K^{\alpha\beta} = \epsilon {}^m K^{\beta\alpha}, \quad {}^m K_{\alpha\beta} = \epsilon {}^m K_{\beta\alpha} \quad (for \ m = even) \quad (54)$$

$${}^m K^{\alpha\beta} = -\epsilon {}^m K^{\beta\alpha}, \quad {}^m K_{\alpha\beta} = -\epsilon {}^m K_{\beta\alpha} \quad (\text{for } m = \text{odd}) \quad (55)$$

The set of operators ${}^m K_\beta^\alpha$, ${}^m K^{\alpha\beta}$, ${}^m K_{\alpha\beta}$ constructed by recursion in the HNN papers has tensor properties that depend only on the parity of m . This property, pointed out only by Nishiyama, is however valid also for the set of operators constructed by Hwa and Nuyts.

Identification of the kinematical constraints

Nishiyama (1976) proved that all kinematical constraints which intervene in Yamamura's (1974) algebraic approach to the nuclear collective motion can be obtained from the following equations of second degree:

$${}^2 K_\beta^\alpha = \frac{1}{4}(2n-1)\delta_\beta^\alpha, \quad {}^2 K^{\alpha\beta} = {}^2 K_{\alpha\beta} = 0 \quad (56)$$

Taking into account Eqs.(48) and (50) the last two relations become

$${}^2 K^{ij} = 2([A_{ik}, B_{kj}]_+ - [B_{ik}, A_{jk}]_+) = 2((AB - BA^t) + (AB - BA^t)^t)_{ij} = 0 \quad (57)$$

$${}^2 K_{ij} = 2([C_{ik}, A_{kj}]_+ - [A_{ki}, C_{kj}]_+) = 2((CA - A^t C) + (CA - A^t C)^t)_{ij} = 0 \quad (58)$$

which are precisely the first two equations (40),(41) written for $\epsilon = +1$, i.e. for the orthogonal algebra, considered in Nishiyama's paper.

Finally, taking again into account Eqs. (48) and (50), the first equation becomes

$${}^2 K_j^i = \frac{1}{2}([A_{ik}, A_{kj}]_+ - [B_{ik}, C_{kj}]_+) = \frac{1}{4}(2n-1)\delta_j^i \quad (59)$$

whence

$$A^2 - BC + ((A^t)^2 - CB)^t = \frac{2n-1}{2} \quad (60)$$

which coincides with Eq. (42).

We shall prove, in the following, that the HNN recursion properties admit a simple and simultaneous proof. To do that, we note that the irreducible tensors in $U(L)$ are obtained by symmetrization from the irreducible tensors in $S(L)$ that transform under the same irreducible representation; it is therefore sufficient to prove the recursion properties for the tensors in $S(L)$.

4 Matrix form of the identities

We start by reminding a procedure imagined by Hannabuss [12], and extensively applied by Okubo [13], for the derivation of equations satisfied by the generators of a *given* irreducible representation (IR) λ of a Lie algebra L .

Consider a pair (λ, μ) of IRs of L and the operator

$$O_{\lambda\mu} = \sum_{i=1}^{\dim L} \lambda(e_i) \otimes \mu(e^i) \quad (61)$$

where e_i are elements of a basis in L and e^i are their dual elements with respect to the Killing-Cartan bilinear form $(e_i, e^j) = \delta_i^j$. The operator $O_{\lambda\mu}$ commutes with $\lambda \otimes \mu$ and satisfies [12] its minimal polynomial:

$$P(O_{\lambda\mu}) = \prod_{\omega \in \Omega(\lambda, \mu)} [O_{\lambda\mu} - \frac{1}{2}((\omega + 2\delta, \omega) - (\lambda + 2\delta, \lambda) - (\mu + 2\delta, \mu))I] = 0 \quad (62)$$

where $\Omega(\lambda, \mu)$ is the set of distinct weights in the Clebsch-Gordan series of the product $\lambda \otimes \mu$ and $(\lambda + 2\delta, \lambda)$ is the eigenvalue of the Casimir element for the IR λ . The degree of the minimal polynomial P is thus equal to the number of distinct IRs into which $\lambda \otimes \mu$ decomposes. In particular, *second-degree* polynomials P correspond to Kronecker products $\lambda \otimes \mu$ that decompose in *two* irreducible terms. The matrix elements of the equality $P(O_{\lambda\mu}) = 0$ with respect to one element of the pair (λ, μ) , μ say, lead to an equation satisfied by the generators of the IR λ ; the degree of this equation will be the degree of P .

Let us turn back to the equation $T_\Lambda = 0$ associated with the symmetric tensor T_Λ (of highest weight Λ) of $U(L)$ derived by projection (cf. Sec.2) and determine the IRs λ for which T_Λ vanishes (i.e. find the modules λ annihilated by T_Λ).

We can now look for a convenient "partner" μ of λ such that the Hannabuss procedure produces an equation of second degree i.e let us find an IR μ such that $\lambda \otimes \mu$ decomposes in two irreducible terms and determine this equation. This equation has been proved in [6] to be precisely $T_\Lambda(\lambda) = 0$.

The two ways of finding annihilators (i.e. by projection and by the Hannabuss method) have been proved to be consistent for tensors of second degree [6]. The combination of both methods allows to write the set of equations $T_\Lambda = 0$ in a compact and transparent form, as *matrix equations*; these will allow simple proofs for the recursion relations derived by Hwa and Nuyts [3] and by Nishiyama [2].

As already pointed out at the end of Sec.2, Eqs.(40-42) are verified for $so(2n)$ (for $sp(2n)$) only if A, B, C are generators of the IRs $(m\Lambda_{n-1}), (m\Lambda_n)$ (of the IRs $(m\Lambda_n)$). It has been proved [6] that, for the algebra $so(2n)$, only the Kronecker products of $(m\Lambda_{n-1})$ and $(m\Lambda_n)$ with (Λ_1) have Clebsch-Gordan series of length two; similarly, for $sp(2n)$ only the products $(m\Lambda_n) \otimes (\Lambda_1)$ have two irreducible components: in both cases there is only one "partner" of the "solutions" of the Eqs. (40-42): the fundamental representation (Λ_1) .

A classical analogue of the operator $O_{\lambda\mu}$.

Let us find now a matrix form for the "classical" relations (35)-(37).

To do that, we replace in Eq.(61) the set of generators of one of the linear representations of L , $\mu(e_i)$ say, by a set of functions that generate a Poisson bracket realisation of L on a symplectic G -manifold M ($L = Lie(G)$):

$$e_i \in L \rightarrow f_{e_i}(p) \in C^\infty(M) \quad (p \in M) \quad (63)$$

with the property

$$f_{Ad(g)e_i}(p) = f_{e_i}(g^{-1}p)(g \in G) \quad (64)$$

Define now the mapping mapping

$$K_\lambda : M \rightarrow End V_\lambda \quad (65)$$

(where V_λ is a λ -module) by

$$K_\lambda : p \in M \mapsto K_\lambda(p) \equiv \sum_{i=1}^{dim L} f_{e_i}(p) \otimes \lambda(e^i) \quad (66)$$

$K_\lambda(p)$ is a matrix whose elements are functions that generate a Poisson bracket (PB) realisation of L .

Denoting by $g \mapsto \lambda(g) \in End(V_\lambda)$ the representation of G acting in V_λ , the operator K_λ is G -equivariant; denoting by the same letter λ the representations of $L = Lie(G)$ and of G this means that:

$$K_\lambda(g.p) = \lambda(g^{-1})K_\lambda(p)\lambda(g)(g \in G) \quad (67)$$

Any polynomial relation $P(K_\lambda) = 0$ is also G -equivariant: it results that the matrix elements of the polynomial relations satisfied by K_λ are polynomial relations satisfied by the generators of the PB realisation.

As already mentioned, the symmetrisation of these "classical" polynomial relations gives the "quantum" polynomial relations, i.e. the relations satisfied by the linear representations. The elements of both realisations transform as tensors with the same highest weight and the corresponding extensions of the adjoint action possesses in both cases the same dimension.

Identification of the recursive relations

Let (Λ_1) be the defining representation of $sp(2n, R)$ or of $o(2n, R)$ and let A_{ij} , B_{ij} and C_{ij} , $(i, j = 1, \dots, n)$ be the generators of a PB realisation of these algebras. The associated K_{Λ_1} -mapping has the expression $K_{\Lambda_1} = \begin{pmatrix} A_1 & B_1 \\ -C_1 & -A_1^t \end{pmatrix}$ For the algebras C_n the matrix K_{Λ_1} has $N(N+1)/2$ distinct matrix elements; for the algebras D_n , K_{Λ_1} has $N(N-1)/2$ distinct matrix elements ($N = 2n$). The number of

distinct matrix elements is, for each algebra, equal to the dimension of the corresponding adjoint representation: for C_n we have $\dim(ad) = \dim(2\Lambda_1) = \frac{N(N+1)}{2}$ and, for D_n , $\dim(ad) = \dim(\Lambda_2) = \frac{N(N-1)}{2}$.

The set of "tensorial identities" (35-38) is equivalent with the matrix equation

$$K_{\Lambda_1}^2 = \begin{pmatrix} A_1^2 - B_1 C_1 & A_1 B_1 - B_1 A_1^t \\ -C_1 A_1 + A_1^t C_1 & -C_1 B_1 + (A_1^t)^t \end{pmatrix} = \begin{pmatrix} \frac{I_2}{2n} & 0 \\ 0 & \frac{I_2}{2n} \end{pmatrix} \quad (68)$$

Let us denote

$$A_2 = A_1^2 - B_1 C_1, \quad B_2 = A_1 B_1 - B_1 A_1^t, \quad -C_2 = -C_1 A_1 + A_1^t C_1 \quad (69)$$

Reminding Eqs.(7), we get

$$B_2^t = \epsilon B_2, \quad C_2^t = \epsilon C_2 \quad (70)$$

Thus, in contrast to the submatrices B_1 and C_1 (cf.(7)), the submatrices B_2 and C_2 are antisymmetric for the algebras of type C_n and symmetric for the algebras of type D_n ; this phenomenon is general: odd powers of K_{Λ_1} behave like K_{Λ_1} , i.e. their matrix elements are tensors of type $(2\Lambda_1)$ for algebras C_n and of type (Λ_2) for algebras D_n .

For the $K_{\Lambda_1}^2$ matrix the submatrices are antisymmetric for C_n algebras and symmetric for D_n algebras (cf. Eqs.(70)). Hence, the number of independent matrix elements in $K_{\Lambda_1}^2$ is $N(N-1)/2$ for C_n and $N(N+1)/2$ for D_n . This number of elements of $K_{\Lambda_1}^2$ is equal to $\dim((0) \oplus (\Lambda_2))$ for the algebra C_n and to $\dim((0) \oplus (2\Lambda_1))$ for the algebra D_n . Indeed

$$\dim(0) + \dim(\Lambda_2) = 1 + \frac{(N+1)(N-2)}{2} = \frac{N(N-1)}{2} \quad \text{for } C_n$$

and

$$\dim(0) + \dim(2\Lambda_1) = 1 + \frac{(N-1)(N+2)}{2} = \frac{N(N+1)}{2} \quad \text{for } D_n$$

For all even powers the matrices behave like $K_{\Lambda_1}^2$. This general statement will be proved now by induction.

Proposition Let K_1 be the matrix

$$K_1 \equiv \begin{pmatrix} K_{1,A} & K_{1,B} \\ K_{1,C} & K_{1,D} \end{pmatrix} \equiv \begin{pmatrix} A_1 & B_1 \\ -C_1 & D_1 \end{pmatrix} \quad (71)$$

whose elements satisfy the relations (7) i.e. $D_1 = -A_1^t$, $B_1 = -\epsilon B_1^t$, $C_1 = -\epsilon C_1^t$

The elements of the matrix

$$K_m \equiv (K_1)^m \equiv \begin{pmatrix} K_{m,A} & K_{m,B} \\ K_{m,C} & K_{m,D} \end{pmatrix} \equiv \begin{pmatrix} A_m & B_m \\ -C_m & -A_m^t \end{pmatrix} \quad (72)$$

satisfy the relations

$$D_m = (-1)^m A_m^t, \quad B_m = (-1)^m \epsilon B_m^t, \quad C_m = (-1)^m \epsilon C_m^t \quad (73)$$

Proof. Assume that Eqs. (72), (73) are true. We have to prove that the elements of the matrix $K_{m+1} = (K_1)^{m+1} = K_1 K_m = K_m K_1$ have the same property.

Because $A_1, B_1, C_1, A_m, B_m, C_m$ belong to the symmetric algebra, (for each of the algebras under consideration), we have

$$K_{m+1} = (K_1)^{m+1} = K_1 K_m = K_m K_1 \quad (74)$$

$$K_{m+1,A}^t = (K_1 K_m)_A^t = (K_m K_1)_A^t = A_m^t A_1^t - C_m^t B_1^t \quad (75)$$

and

$$K_{m+1,D} = (K_m K_1)_D = -C_m B_1 + D_m D_1 = (-1)^{m+1} (A_m^t A_1^t - C_m^t B_1^t) \quad (76)$$

and the first equality (74) is proved. Similarly

$$K_{m+1,B}^t \equiv (K_1 K_m)_B^t = (A_1 B_m + B_1 D_m)^t = (-1)^{m+1} \epsilon (A_m B_1 - B_m A_1^t) \quad (77)$$

and

$$K_{m+1,B} \equiv (K_m K_1)_B^t = (A_m B_1 + B_m D_1) = A_m B_1 - B_m A_1^t \quad (78)$$

this proves the second equality (74). The third equality results in the same way.

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